

p-Adic Field Theory limit of TGD is free of UV divergences

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Abstract

The p-adic description of Higgs mechanism in TGD framework provides excellent predictions for elementary particle and hadrons masses (hep-th@xxx.lanl.gov 9410058-62). The gauge group of TGD is just the gauge group of the standard model so that it makes sense to study the p-adic counterpart of the standard model as a candidate for low energy effective theory. Momentum eigen states can be constructed purely number theoretically and the infrared cutoff implied by the finite size of the convergence cube of p-adic square root function leads to momentum discretization. Discretization solves ultraviolet problems: the number of momentum states associated with a fixed value of the propagator expression in the loop is integer and has p-adic norm not larger than one so that the contribution of loop momentum squared with p-adic norm p^k converges as p^{-2k-2} for boson loop. The existence of the action exponential forces number theoretically the decomposition of action into free and interacting parts. The free part is of order $O(p^0)$ and must vanish (and does so by equations of motion) and interaction part is at most of order $O(\sqrt{p})$ p-adically. p-Adic coupling constants are of form $g\sqrt{p}$: their real counterparts are obtained by canonical identification between p-adic and real numbers. The discretized version of Feynmann rules of real theory should give S-matrix elements but Feynmann rules guarantee unitarity in formal sense only. The unexpected result is the upper bound $L_p = L_0/\sqrt{p}$ ($L_0 \sim 10^3\sqrt{G}$) for the size of p-adic convergence cube from the cancellation of infrared divergences so that p-adic field theory doesn't make sense above length scale L_p .

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1 Introduction

The description of Higgs mechanism in TGD framework provides excellent understanding of particle masses [Pitkänen_c]. The cornerstones of the approach are following:

- a) The existence of p-adic square root in the vicinity of p-adic real axis implies four-dimensional algebraic extension of p-adic numbers identifiable locally as p-adic spacetime. p-Adic version of conformal invariance is suggested both by the criticality of TGD:ish Universe at quantum level as well as the existence of minima of Kähler action, which are p-adically analytic maps from p-adic M^4 to p-adic CP_2 in flat space approximation [Pitkänen_b]. The commutators of infinitesimal conformal symmetries with $N = 1$ supersymmetry generated by the right handed neutrino plus related kappa symmetry extend the conformal invariance to super conformal invariance.
- b) Super conformal invariance together with basic assumptions of TGD leads to a unique identification of elementary fermions and bosons as tensor products of representations of p-adic Super Virasoro algebra (Kac Moody spinors).
- c) The support of p-adic square root function in the vicinity of p-adic real axis can be regarded as p-adic version of light cone and consists of convergence cubes (rather than spheres) of p-adic square root function (see the first paper of [Pitkänen_c]). This suggest that the construction of p-adic conformal field theory limit should reduce to the construction of n-point functions for conformal field theory defined on convergence cube of square root function.
- d) p-Adic convergence cube can be regarded as a particle like object in length scales above the size of convergence cube and the suggested rough formulation of p-adic conformal field theory limit in the first paper of [Pitkänen_c] was based on this particle concept. The mass calculations however demonstrate that in good approximation particles can be regarded as boundary components so that the 'points' of n-point functions correspond to boundary components of p-adic convergence cube. In point particle limit it does not matter whether the boundaries of small topologically condensed 3-surfaces or the boundaries of small holes drilled on the background surface are in question. If this the case then conformal field theory treating boundaries (of, say, holes drilled on p-adic convergence cube) as point like objects described by Kac Moody spinors inside p-adic convergence cube should provide an excellent description of particle physics phenomena. Even more, since Planck

mass excitations are expected to have small effect on low energy physics and the gauge group of TGD is the gauge group of standard model the p-adic version of standard model might provide good approximation for the theory.

The hard part of the job is the explicit construction of the conformal field theory for Kac Moody spinors inside the convergence cube. This requires the generalization of ordinary gauge field theory defined for finite component fields to a gauge field theory for infinite component fields provided by Kac Moody spinors. The main technical problem seems to be the construction of vertices: what is required is to find Super Virasoro invariant action of a state in N-S type representation on N-S or Ramond type representation. A related task is to formulate general conformal field theory limit using p-adic version of Kähler Dirac action used to define configuration space metric and spinor structure and to show that super conformal invariance indeed results and that the Kac- Moody spinor concept developed during the mass calculations emerges naturally from this formalism. The work related to both these problems is in progress.

In this paper a more modest approach is adopted. p-Adic thermodynamics predicts low energy mass spectrum with excellent accuracy and gauge group is just standard model gauge group. Therefore a good guess is that p-adic YM theory more or less identical with standard model (without Higgs field) should provide a good approximation of p-adic conformal field theory at non Planckian energies. In this paper the general conceptual framework necessary for the construction and physical interpretation of the theory is studied.

- i) The relationship between p-adic and real unitarity and probability concepts makes possible the physical interpretation of the theory. Some strikingly new effects are predicted (see the fifth paper of [Pitkänen_c]).
- ii) The construction of momentum eigenstates as p-adic planewaves involves elegant number theory and as predicts number theoretic mechanism for generation of new physically interesting length scales.
- iii) Discretization of momenta by the necessary infrared cutoff implied by number theory and finite size of convergence cube implying automatically the absence of ultraviolet divergences.
- iv) Perturbation theory has purely number theoretic justification. Free theory corresponds to $O(p^0)$ sector and the existence of the action exponential requires the vanishing of free field action. Same requirement implies that

interactions to higher powers of \sqrt{p} . In particular, the effective values of p-adic gauge couplings are effectively of the form $g\sqrt{p}$, where g is rational number and their real counterparts obtained by canonical identification are indeed what they should be.

v) At tree level the predictions of the theory does not seem to deviate very much from the predictions of ordinary field theory. The general features of the coupling constant evolution, in particular breaking of QCD perturbation theory at length scale $L_p = \sqrt{p}L_0$ (of order hadronic length scale) can be understood number theoretically. In fact, all elementary particles can appear as quantum states only below the length scale L_p in accordance with the TGD result that 3-surfaces associated with charged particles have finite size for topological reasons [Pitkänen_b]. Therefore field theory description using elementary particles as basic dynamical objects doesn't seem to work above the length scale L_p .

The results concerning the concept of p-adic planewave and the absence of UV divergences are expected to generalize as such to the more general gauge theory formulated for Kac Moody spinors and also to the p-adic conformal field theory formulated in terms of Kähler Dirac action.

2 p-Adic unitarity and probability concepts

p-Adic unitarity and probability concepts discussed in [Pitkänen_b] lead to highly nontrivial conclusions concerning the general structure of S-matrix. S-matrix can be expressed as

$$\begin{aligned} S &= 1 + i\sqrt{p}T \\ T &= O(p^0) \end{aligned} \tag{1}$$

for $p \bmod 4 = 3$ allowing imaginary unit in its four-dimensional algebraic extension. Using the form $S = 1 + iT$, $T = O(p^0)$ one would obtain in general transition rates of order inverse of Planck mass and theory would have nothing to do with reality. Unitarity requirement implies iterative expansion of T in powers of p (see fifth paper of [Pitkänen_c]) and the few lowest powers of p give extremely good approximation for physically interesting values of p .

The relationship between p-adic and real probabilities involves the hypothesis (for details see the fifth paper of [Pitkänen_c]) that transition probabilities depend on the experimental resolution. Experimental resolution is defined by the decomposition of the state space H into direct sum $H = \oplus H_i$ so that experimental situation cannot differentiate between different states inside H_i . To each resolutions there are associated different real transition probabilities unlike in ordinary quantum mechanics. Physically this means that the experimental arrangements, where one monitors each state in H_i separately differ from the situation, when one only looks whether the state belongs to H_i . One application is related to momentum space resolution dependence of transition probabilities. More exotic application described in the fifth paper of [Pitkänen_c] is related to Z^0 decay widths: the total annihilation rate to exotic lepton pairs (unmonitored) is essentially zero: if one would (could) monitor each exotic lepton pair one would obtain simply sum of the rates to each pair.

3 p-Adic planewaves

The definition of p-adic momentum eigen states is a nontrivial problem. The point is that usual exponent function $f_P(x) = \exp(iPx)$ does not make sense as a representation of momentum eigen state. f_P is not periodic function, f_P does not even converge if the norm of Px is not smaller than one and the orthogonalization of different momentum eigen states is not possible. For instance, the sum f_P over discretized argument x does not in general vanish since lowest order contribution is just the number of points x . This state of affairs suggests that p-adic momentum concept involves number theory. It turns out that this is the case and that momentum space has natural fractal structure.

3.1 The concept of primitive root

The fundamental requirement for planewave is periodicity. If the size of the p-adic converge cube, assumed for simplicity to be one-dimensional, is $L_p = \sqrt{p}L_0$ then there exists an elegant manner to define planewaves. Ultraviolet cutoff at $L_0 \sim 10^3\sqrt{G}$ implies discretization in x-space and one can label the points of cube by numbers $x = 0, .1, ..., p-1$: it turns out that this restriction

is not in fact necessary but simplifies the argument. The basic observation is that p-adic numbers allow roots $a = a_0 + a_1p + \dots$ of unity satisfying the condition $a^n = 1$ for some values of n . The condition reduces in lowest order to the condition $a_0^n = 1 \bmod p$. One can interpret a_0 as an element of finite field $G(p, 1)$ [Schroeder]. The obvious idea to use p :th root of unity and its powers to define planewave basis containing p states. p :th root of unity do not exist however unless one performs extension of p-adic numbers. $p - 1$:th root however exists and following facts hold true [Schroeder].

- a) $a_0^{p-1} \bmod p = 1$ is identically satisfied for any a_0 in $G(p, 1)$. From this it follows that the order n of a_0 is always factor of $p - 1$ so that only finite number of orders ($< p$) are possible for a_0 and also for a .
- b) For the so called primitive roots allowed by any prime the order is maximal: $n = p - 1$. If m does not divide $p - 1$ then also a_0^m is primitive root.
- c) The number of roots for arbitrary integer n is given by $\Phi(n)$ defined as the number of integers $k < n$ not dividing n , $k = 1$ included. For $n = p$ one has clearly $\Phi(p) = p - 1$ corresponding to numbers $a_0 = 1, \dots, p - 1$.

3.2 p-Adic planewaves with momenta $k = 0, \dots, p - 1$ and number theoretic generation of length scales

What comes first into mind is to define plane waves as functions

$$f_k(x) = a^{kx}, k = 0, 1, \dots, p - 1 \quad (2)$$

where a is some p-adic primitive root of 1 modulo p and k is an integer running from $k = 0$ to $p - 1$. There are $\Phi(p) = p - 1$ different plane waves with this definition and this looks problematic since p planewaves are expected on physical grounds. The lacking state should obviously correspond to momentum $k = p - 1$ and indeed does so. The point is that this state is not identical with $k = 0$ state p-adically as suggested by $a^{p-1} = 1$. This can be seen by considering $k = p - 1$ planewave at points of form $x = y/(p - 1)$.

The conjugate of the p-adic planewave is just a^{-kx} , which is well defined in $G(p, 1)$ as well as p-adically. The sum of $f_k(x)$ over $x = 0, 1, \dots, p - 1$ vanishes modulo p and also p-adically. This follows from the decomposition of the polynomial $\sum_0^{p-1} z^k$ to product of terms $z - a^k$, where a is primitive root. This means that orthonormalization modulo p is guaranteed. In practice

x-space discretization does not matter since p-adic field theory limit applies only at length scales above the cutoff scale of order 10^3 times Planck length [Pitkänen_b].

The natural identification of the real counterpart of momentum P is as integer proportional to k : $P/2\pi = k$. In fact, it is $P/2\pi$, which appears in all formulas of p-adic Higgs mechanism rather than P so that the p-adic nonexistence does not produce problems. The real counterparts of the momenta can be defined by adding the factor of 2π to the real counterpart of p-adic momentum. Momentum does not correspond directly to the inverse of the wavelength as in real context. The wavelength λ is just the order n of the element a^k and is a factor of $p - 1$ and the degeneracy associated with a given factor n is $\Phi(n)$.

One might wonder whether this selection of possible wavelengths has some physical consequences. The average value of prime divisors counted with the degeneracy of divisor is given by $\Omega(n) = \ln(\ln(n)) + 1.0346$ [Schroeder] and is suprisingly small, or order 6 for numbers of order M_{127} ! If one can apply probabilistic arguments or [Schroeder] to the numbers of form $p - 1$, too then one must conclude that very few wavelengths are possible for general prime p ! This in turn means that to each p there are associated only very few characteristic length scales, which are predictable. Furthermore, all the p^k -multiples of these scales are also possible if p-adic fractality holds true in macroscopic length scales.

Mersenne primes M_n can be considered as an illustrative example of the phenomenon. From [Brillhart *et al*] one finds that $M_{127} - 1$ has 11 distinct prime factors and 3 and 7 occurs three and 2 times respectively. The number of distinct length scales is $3 \cdot 2^{11} - 1 \sim 2^{12}$. $M_{107} - 1$ and $M_{89} - 1$ have 7 and 11 singly occurring factors so that the numbers of length scales are $2^7 - 1 = 127 = M_7$ and $2^{11} - 1$. Note that for hadrons (M_{107}) the number of possible wavelengths is especially small: does this have something to do with the collective behaviour of color confined quarks and gluons? An interesting possibility is that length scale generation mechanism works even macroscopically (for p-adic length scale hypothesis at macroscopic length scales see [Pitkänen_b]). Long wavelength photons, gravitons and neutrinos might therefore provide a completely new mechanism for generating periodic structures with preferred sizes of period.

3.3 Fractal construction of p-adic planewaves with higher momenta

Particle in a box picture suggests that momentum spectrum indeed possesses infrared cutoff but that it should be possible to realize all momenta $k = np^{-k}$ for $k \geq 0$.

a) Consider first momenta $k = n/p$ with p-adic norm p . The plane wave formula can be generalized by writing

$$\begin{aligned} f_{p^{-r}n}(x) &= a^{p^{-r}nx} = f_n(p^{-r}x) \\ |x|_p &\leq p^{-r} \\ r &= 1 \end{aligned} \tag{3}$$

This function however exists for x having norm not larger than p^{-1} so that the state is localized and can be regarded as momentum eigenstate in the length scale defined by the support of the planewave, only.

b) These planewaves are certainly not all what is needed since the functions representable in the basis could have arbitrary large gradients only in the immediate vicinity of $x = 0$. One can however translate the plane waves located around origin $x_a = 0$ to all points $x_a = n$, $n = 0, p-1$, which means the replacement $x \rightarrow x - x_a$ in the previous formula. In this manner one obtains altogether $N(r=1) = p(p-1)$ localized planewaves with p-adic momenta $p^{-1}k$ since constant planewave is excluded by infrared cutoff.

c) The construction of p-adic plane waves with p-adic momenta with p-adic norm p^r proceeds in obvious manner. One constructs around each point

$$x_a = \sum_{k=0}^{r-1} x_k p^k \tag{4}$$

a localized planewave basis

$$f_{p^{-r}k}(x - x_a) \tag{5}$$

with p-adic momenta $p^{-r}k$ and argument $x - x_a$ and obtains in this manner p^{r+1} states.

The localization of planewaves is not in conflict with Uncertainty Principle since the localized planewaves are momentum eigenstates only in the length scale defined by the support of localized state: this is also clear from the fact that momentum spectrum contains only the momenta $P = p^{-r}k$ but not the more general momenta $P = \sum_r k_r p^{-r}$. The number of localized momentum eigenstates with the p-adic norm of momentum not larger than p^r is $p^{r-1}(p-1)$, where the factor $p-1$ comes from infrared cutoff excluding constant planewave, and indeed equal to the possible values of momenta. Infrared cutoff in momentum is necessary. One can obviously construct genuine momentum eigenstates simply as products of momentum eigenstates associated with different length scales. If one tries to extend the momentum range to infrared one encounters problems with completeness of the basis since p-adic convergence cube does not contain the entire range of x values for which the plane wave is well defined (Uncertainty Principle!).

Momentum space has fractal structure, the number of momenta with p-adic norm p^r being $N(r) = p^{r-1}(p-1)$ in single spatial dimension. In the limit of infinite UV cutoff the total number of states in for dimension D is just

$$N(tot, r) = p^{D(r-1)}(p-1)^D \quad (6)$$

If the total number of states is defined by taking ultraviolet cutoff to infinity the number of states is p-adically equivalent with zero: a somewhat unexpected result! The result is important as far as vacuum expectation values and normal ordering of oscillator operators is considered: for instance, the normal ordering of fermion current gives no c-number term.

One can generalize the planewave concept somewhat. Since the algebraic extension used allows square root one can define new planewave basis as square roots of the p-adic planewaves: the corresponding momentum spectrum obviously contains half odd integers. These planewaves are in general genuinely complex p-adic numbers. The preliminary work with conformal field theory limit suggests that the existence of two kinds of plane waves is directly related to the existence of Ramond and N-S type representations and that the momentum spectrum for Ramond/N-S type super generators is labeled by Z integers and by $Z/2$. Furthermore, lepton and quark momenta should belong to Z ($Z/2$) respectively.

4 Second quantized interacting field theory inside p-adic convergence cube

In practice YM theory with standard model gauge group for leptons and quarks plus the some other light exotic particles predicted by the p-adic thermodynamics should provide excellent description of the physics below non-Planck energies. The extremely rapid convergence of the p-adic perturbation theory implies that loop corrections coming from Planck mass excitations are extremely small for physical values of prime p and can be neglected. The task is to find whether it is possible to define second quantized interacting theory inside single convergence cube so that S-matrix has the structure dictated by physicality requirements.

4.1 Decomposition of action into free and interacting parts number theoretically

The requirement that perturbation theory works requires that interaction terms V are proportional to the factor \sqrt{p} or some higher power of p whereas free part of the action is of order $O(p^0)$ and must vanish, not only by field equations, but also by the requirement that action exponential exists p-adically. One could use functional integral formalism or Hamiltonian approach and in both of these approaches same number theoretic constraint is encountered.

In functional integral formalism one considers the exponent of classical action. Kinetic term of the action is of order $O(p^0)$ formally. The exponent of the action makes no sense unless kinetic term vanishes identically: this in turn is in accordance with equations of motion of free field theory in order $O(p^0)$ provided action vanishes for free field solutions. This in turn gives strong constraint on the action: for instance, the generation of cosmological constant via vacuum energy density becomes impossible. Interaction term in turn must be of order $O(\sqrt{p})$ at least so that S-matrix has the required form and exponent of action exists. Functional integration is over over quantum fluctuations around classical solution with vanishing action and one must require that integration is only over quantum fluctuations, whose contribution to the action of order $O(p^{1/2})$ at most.

In Hamiltonian formulation one expresses the solutions as M^4 fields perturbatively using time ordered exponential $P(\exp(i \int V dt))$, where V is $O(\sqrt{p})$ contribution of Hamiltonian. V must be proportional to \sqrt{p} for the integral to exist. The existence of $P(\exp(\int V dt))$ probably poses an upper bound for the transition time T (size of the convergence cube in time direction) since the exponential is not expected to exist for too large values of T . This means that quantum transition times are naturally quantized. The value of this time is naturally the duration of time associated with single p-adic convergence cube. There is no particular reason to expect that all cube sizes are possible and it turns out that p-adic counterpart of standard model does not exist in length scales above L_p .

These arguments suggest that perturbative approach is the only manner to define p-adic quantum theory. The fields are expressible in terms of free part Φ_0 and interacting part

$$\Phi = \Phi_0 + \sum_{n \geq 1} \sqrt{p}^n \Phi_n \quad (7)$$

The free field Φ_0 has standard expansion in terms of oscillator operators in one-one correspondence with light states associated with ordinary spinors and gauge fields. Φ_n contains off mass shell momenta and can be solved iteratively in terms of Φ_0 using the equations of motion.

One can define the oscillator operators of the interacting theory as power series expansions and calculate S-matrix. Direct manner to construct S-matrix is LSZ reduction formula applied inside p-adic convergence cube. What one obtains is QFT in box determined by the convergence cube of square root with infrared cutoff coming from the size of the cube. A reasonable guess is that the Feynmann rules of standard gauge theory apply as such since all algebraic manipulations of standard gauge theory go through as such: situation is even simpler since the elimination of ultraviolet divergences is not needed. p-Adic unitarity is guaranteed in formal sense by Feynmann rules but the necessary infrared cutoff might lead to problems with unitarity.

4.2 Action and Feynmann rules

The ordinary YM Dirac action should describe the couplings of the nonexotic light states. The couplings associated with vertices containing exotic states

contain yet unknown parameters, which are predicted by p-adic conformal field theory having as its particle content single particle states of second quantized theory. The masses can be taken to be the masses predicted in excellent approximation by p-adic thermodynamics. CKM mixing matrix appear in quark couplings and it was found the requirement that topological mixing matrices are rational unitary matrices together with some other TGD:ish requirements might well determine these parameters uniquely [Pitkänen_c].

The assumption that p-adic gauge couplings have the general form

$$g_p = g\sqrt{p} \tag{8}$$

guarantees $V \propto \sqrt{p}$. This definition of effective coupling was suggested in the fifth paper of [Pitkänen_c] and certainly provides a correct relationship between p-adic and real coupling. If g is rational number then the real counterpart $(g^2 p)_R$ of $g^2 p$ obtained by canonical correspondence between p-adics and reals is reasonably close to g^2 interpreted as ordinary rational number. In particular, for Mersenne primes and g^2 , which is finite superposition of negative power of 2 the $(g^2 p)_R$ is numerically very near to g^2 interpreted as real number. This implies for YM theory that at tree graph level each internal line contains two \sqrt{p} :s at its ends and for loop momenta between elementary particle mass scale and Planck mass scale one p in propagator so that p :s cancel and one obtains something very near to that of ordinary gauge theory.

The naivest definition of the vertices in perturbation theory is not however the most elegant one. One can always redefine gauge potentials so that gauge couplings appears nowhere except as a normalization factor $\frac{1}{4g^2 p}$ of gauge boson part of the YM action. This definition in turn implies that gauge boson propagators are proportional to $g^2 p$ and fermion boson vertices involve no coupling constant. The use of this description is suggested also by the concept of induced gauge field, which is naturally such that gauge couplings is included into the normalization of gauge potentials. Of course, this definition doesn't change the definition of, say, effective α_s but makes it easy to see that various lowest order loop corrections are p-adically of same order of magnitude below L_p .

The definition of fermionic propagators is not quite straightforward since the quantity

$$\begin{aligned}
G &= \frac{1}{p_k \gamma^k + M_{op}} \\
M_{op}^2 &= M^2
\end{aligned} \tag{9}$$

involves the square root M_{op} of mass squared of fermion calculated in the papers [Pitkänen_c] as thermal expectation value. Dirac operator acts on 8-dimensional H-spinors and must respect chirality conservation. Therefore M_{op} cannot be scalar but rather a linear combination $a\gamma_0 + b\gamma_3$ of CP_2 tangent space gamma matrices commuting with electromagnetic charge operator. M_{op} is clearly the counterpart of Higgs vacuum expectation value and the geometric counterpart of M_{op} is the CP_2 part of the operator $H^k \gamma_k$ appearing in the Dirac equation for induced spinors, H^k being the trace of the second fundamental form. When boundary components are idealized to world lines one can assume that $H^k(CP_2)\gamma_k$ is covariantly constant and $H^k(M_+^4)\gamma_k$ vanishes (geodesic motion in M_+^4). The coefficients a and b are different for different charge states of lepton/quark so that it is possible to find a solution to the defining condition although the solution need not be unique.

The essential difference with respect to ordinary gauge theory is the number theoretic description of momentum eigenstates and discretization of momenta. The discretization solves also the problem implied by the p-adical nonexistence of standard momentum space measure $dV = (2\pi)^3 d^3p / 2E_p$ (π does not exist p-adically). p-Adic discretization also implies also the absence of ultraviolet divergences as will be found. The cancellation of infrared divergences implies that the length scale $L_p) = L_0 \sqrt{p}$ gives upper bound for the size of the p-adic convergence cube: for larger size the sum of self energy diagrams is not p-adically convergent.

4.3 How to compare predictions with experiment

The comparison of the theory with ordinary field theory and experiments is based on the concept of resolution. In momentum degrees of freedom this means that p-adic momentum space is divided into cells so that different momenta inside cells are not monitored experimentally and the summation over final states is performed p-adically. The resulting transition amplitudes

squared are mapped to reals by canonical identification and after this the usual momentum space integration measure can be used. In practice, the transition rates are the physically interesting quantities. In standard QFT the squares of S-matrix elements involve square of momentum conserving delta function and the rate is obtained by dividing with the momentum delta function interpreted as infinite quantization volume. In present case the quantization volume is finite and given by the volume of the convergence cube of p-adic square root function.

5 Number theoretic cancellation of UV divergences and necessity of infrared cutoff

L_p

The fact that momentum summation always involves integers with p-adic norm not larger than implies the cancellation of ultraviolet divergences provided the imbedding of M^4 differs slightly from standard imbedding. Infrared cutoff in turn is forced by infrared finiteness. In the following considerations are restricted to scalar loops but similar considerations can be applied fermionic loops and fermion self energies.

5.1 UV finiteness

Consider first the self energy contribution, when bosons propagate in the loop. The summation of all possible loop momenta can be decomposed to sum over terms for which the scalar function

$$F(P_1, P, P_1 - P) = \frac{g^2 p^2}{(P^2 - M_1^2)((P_1 - P)^2 - M^2)} \quad (10)$$

associated with the loop is constant. Here we have assumed that bosonic propagators proportional to p to guarantee that the contributions of fermionic and bosonic loops are of same order. For very large loop momenta the function is in good approximation $g^2 p^2 / P^4$ and for a given p-adic norm p^{-k} of ultraviolet momentum squared behaves as p^{2k+2} p-adically. This implies rapid convergence in ultraviolet since the number of all possible loop momenta is

always integer and has p-adic norm not larger than one. This means that ultraviolet divergences are completely absent!

5.2 The problem of ill defined degeneracy factors and propagator poles

UV finiteness does not yet guarantee the p-adic existence of the momentum sums associated with self energy diagrams. For spacelike incoming propagator momenta one finds that degeneracy factor associated with certain loop momenta is infinite. Although the p-adic norm of this number is not larger than one it is ill defined. Second problem is related to the poles of p-adic propagator, when some of the discrete loop momenta are on mass shell. Both problems can be circumvented by using Wick-rotation trick or by taking into account the deformation of standard imbedding of M^4 to nonstandard but still flat imbedding caused by total gravitational mass of the p-adic convergence cube.

5.2.1 The problem

Suppose that incoming space like propagator momentum P is given by $(0, P, 0, 0)$ and denote by $(0, K, 0, 0) / k_\perp$ the component of loop momentum k in direction parallel/orthogonal to P . Strictly speaking, this P is not allowed by infrared cutoff the situation is essentially equivalent with that obtained assuming all components of P to be nonvanishing. The propagator expression associated with self energy contribution is for scalar particle given by

$$\frac{1}{(-(P-K)^2 - M^2)(-K^2 + k_\perp^2 - M^2)} \quad k_\perp = (k_0, 0, k_1, k_2) \quad (11)$$

The value of propagator expression is fixed once the values of k^2 and $P \cdot k$ are fixed.

Any lightlike vector k_\perp contributes to the degeneracy and there are infinite number of light like vectors of this kind since the equation

$$k_0^2 = k_1^2 + k_2^2 \quad (12)$$

allows infinite number of solutions. The first solution type consists of vectors $(k_0, 0, k_1, 0)$ and $(k_0, 0, 0, k_1)$ and the number solutions is $(p-1)p^N$ $N \rightarrow \infty$: the number of solutions has vanishing p-adic norm at the limit $N \rightarrow \infty$ so that the contribution to self energy vanishes. In fact, this solution is excluded by strongest form of infrared cutoff requiring that each momentum component is nonvanishing.

The second solution type consists of solutions for which k_1 and k_2 are nonvanishing. A short calculation shows that the general solution is of form

$$\begin{aligned} k_i &= (K_0, 0, K_1, K_2) \sum_{n \geq 0} \epsilon_n p^{-n} \\ K_0^2 &= K_1^2 + K_2^2 \\ K_0 &< p \\ \epsilon_n &\in (0, 1, -1) \end{aligned} \tag{13}$$

where K_i is integer solution of the condition. The total number of integer solutions is the number of Pythagorean triangles with hypotenuse smaller than p : integers scaled triangles are counted as different triangles. Denoting the number of these triangles with $N(p)$ one has for the total degeneracy

$$N(tot) = \lim_{N \rightarrow \infty} 3^N N(p) \tag{14}$$

This limit is not well defined p-adically (except in case $p = 3$) although the p-adic norm of limit is well defined. If one allows only non-negative momenta the power of 3 is replaced with power of 2.

5.2.2 Wick rotation solves the problem?

The troubles clearly result from the possibility of light like vectors k_\perp in 3-dimensional hyperplane of momentum space. Therefore Euclidian signature for momentum space implies that loops sums are completely well defined and analytic continuation in P makes it possible to define momentum sums for M signature. Also poles disappear. Wick rotation is routinely applied in ordinary QFT but the application of Wick rotation in present case is somewhat questionable trick.

5.2.3 Gravitational warping solves the problem?

The infinite degeneracy associated with spacelike momenta P disappears if the standard imbedding of M^4 is replaced with warped imbedding, for which some components of metric differ from their standard values by scaling. The phenomenon is completely analogous with the existence of infinitely many imbeddings of flat 2-dimensional space E^2 to E^3 obtained by deforming any plane of E^3 without stretching it. The physical reason for warping is gravitational mass of the p-adic convergence cube. The mass of the p-adic convergence cube deforms spacetime surface and average effect is momentum dependent scaling of some metric components. In GRT gravitational warping clearly does not make any sense. The simplest physical consequence of the deformation is that photons propagating in condensate spend in general longer time than vapour phase photons to travel a given distance [Pitkänen_a].

A rather general nonstandard imbedding of M^4 representing gravitational warping is obtained by assuming that CP_2 projection of spacetime surface is a one-dimensional curve:

$$\begin{aligned} s &= KP_k m^k \\ g_{\alpha\beta} &= m_{\alpha\beta} - K^2 P_\alpha P_\beta \end{aligned} \tag{15}$$

Here s is the curve length coordinate and P_k is total four-momentum.

The scaling of the metric components has two nice consequences:

- a) Assuming that the spectrum of off-mass shell four-momenta is not changed the propagator expression has no poles since one has always $k^2 = g^{\alpha\beta} k_\alpha k_\beta \neq M^2$ in warped metric.
- b) For $m_{00} = 1 \rightarrow 1 - \delta$ the expression $k_\perp^2 = 0$ becomes $k_\perp^2 = -\delta k_0^2$ and this number becomes p-adically very large, when p-adic norm of k_0 increases. This guarantees the convergence of the propagator expression as well as finite degeneracies for all values of P .

5.3 Necessity of infrared cutoff

In p-adic field theory the dangerous region seems to be infrared rather than ultraviolet as far as momentum space summation is considered. Now however number theoretic (and physically obvious) infrared cutoff resulting from the

finite size of convergence comes to rescue and makes self energy finite in lowest order. Although self energy is certainly finite the geometric sum of self energies does not converge unless self energy is of order $O(p)$ and therefore same order as the momentum propagating in the loop if virtual momentum is between particle mass and Planck mass. This condition is not however always satisfied.

a) If the particles propagating in the loop are massive then all contributions to self energy are at most of order $O(p)$ irrespective of the size $\sqrt{p^n}L_0$ of the convergence cube.

b) If loop contains massless particles then situation changes since lowest momentum square in massless loop has p-adic norm p^n . The point is that at worst the massless loop particle gives contribution

$$\frac{1}{P^2} \propto p^{-n} \quad (16)$$

to self energy. The proportionality of boson propagators to p can give at most the power p^k , $k = 1, 2, 3$ so that for sufficiently large n the self energy becomes large and the geometric sum defined by self energy does not converge p-adically. One could formally define the sum as the sum of geometric series but the resulting propagator would contain positive power of p making it extremely small so that this trick provides only a second manner to say that something drastic happens in length scale L_p .

The result means that perturbation theory is simply not well defined in length scales above $L_p = L_0\sqrt{p}$, which is just of order of Compton wave length for primarily condensed particles.

a) Masslessness of photon implies that charged fermions cannot appear as states of quantum field at length scales above L_p in accordance with the basic results of classical TGD.

b) Masslessness of gluons in turn implies that QCD is not well defined above L_{107} : a clear signal of color confinement. It is p-adic cubes with size L_{107} , which become basic dynamical units at these scales and these must be non-colored since otherwise gluon propagators would lead to exactly the same difficulty as in quark level.

c) Since photon self energy at lowest order involves only massive charged particles one might think that photon self energy is always of order $O(p)$ so that photon propagation would be possible in all length scales. The problem

is that the higher loop contributions to self energy contains emission of arbitrary number of virtual photons from charged loop particles and this makes the self energy large in length scales larger than L_p . Same applies also to intermediate gauge bosons.

The general conclusion seems to be that above L_p quantum field theory description at condensation level p fails. It is however possible for a particle to suffer secondary condensation on level $p_1 > p$ and at this level quantum field description is possible below larger scale $L_{p_1} = \sqrt{p_1}$. In this manner one obtains condensation hierarchy. This fits nicely with the results of classical TGD. The electroweak gauge fields in TGD are induced from the spinor connection of CP_2 and for purely topological reasons the imbedding of charged gauge field fails for some critical size of the 3-surface. In particular, the size of 3-surface associated with charged particle and in fact any particle creating long range fields is finite: what was called 'topological field quantum' is formed [Pitkänen_b]. It was this concept, which together with p-adic length scale hypothesis led to the detailed quantitative development of the concept of topological condensate [Pitkänen_a, Pitkänen_b] in various length scales. Note that the results obtained make also particle-field duality concept very concrete.

Of course, one can ask what replaces the QFT description above length scale L_p . One possibility is that L_p represents absolute upper size of particles at level p so that this kind of description is not needed. A second possibility is that elementary particles are replaced with composite objects (say hadrons) as fundamental fields. In practice, it seems the p-adic version of ordinary quantum mechanics might provide satisfactory description above L_p .

5.4 Note about coupling constant evolution

The calculation of the details of coupling constant evolution is not easy since the calculation necessitates the calculation of degeneracies for a given value of propagator expression and this involves difficult number theory. It would not be surprising if the p-adic counterparts typical logarithmic terms of ordinary coupling constant evolution would be encountered.

It is however not obvious what the p-adic counterparts of logarithms of type $\ln(Q^2/Q_0^2)$ are. In the fifth paper of [Pitkänen_c] it was noticed that

logarithm defined as inverse of the ordinary exponent function exists only for $Q^2/Q_0^2 = 1 + O(p)$ so that each range $Q^2 = p^k(n + O(p))$ defines its own coupling constant evolution. The construction of p-adic planewaves suggests a more satisfactory solution of the problem. The correct manner to determine logarithm is start from the exponent a^x of p-adic primitive root and define logarithm via the formula

$$\begin{aligned} y = \log_a(x) &\leftrightarrow x = a^y \\ a^{p^{-1}} &= 1, |x|_p = 1, |y| \geq 1 \end{aligned} \quad (17)$$

Logarithm is well defined for all values of x having p-adic norm equal to one. The value of the logarithm can have arbitrary large p-adic norm not smaller than one. Therefore each power of p in momentum space defines its own coupling constant evolution and coupling constant evolution equations in principle involve the initial values $g^2(k)$ defined as

$$g^2(k) = g^2(Q_0^2 = p^{-k}) \quad (18)$$

In practice, the results about cancellation of infrared divergences suggest that only single value of k is needed and this s power of p involves entire energy range from the size of particle to Planck length.

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